Liquidity Provider Wealth

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Abstract

Some fundamental questions about the behavior of constant product markets, such as Uniswap, have not yet been answered. The expected returns of their Liquidity Providers ("LPs") are of particular interest:

1. What is the expected growth rate of a Uniswap LPs portfolio?
2. What fee maximizes this growth rate?
3. Can the optimal fee's growth rate exceed that of a buy-and-hold portfolio?

1 Introduction

We will first give a definition of constant product markets and walk through a brief derivation of the no-arbitrage conditions that govern their behavior. This formalization is a distilled summary of some results in our previous paper with Angeris et al. [AKC$^+$19].

We will then introduce a price process for the underlying assets of the constant product market (one cash and one risky), a simple geometric Brownian motion. We are not aware of any existing work that incorporates both an evolving price process and a non-zero fee into a dynamical system model. This is necessary to analytically characterize the LP expected returns in practice.

1.1 Constant product markets

A *constant product market* [ZCP18] is a market for trading coins α for coins β (and vice versa). This market has reserves $R_\alpha > 0$ and $R_\beta > 0$, constant product $k = R_\alpha R_\beta$, and percentage fee $(1-\gamma)$. A transaction in this market, trading $\Delta_\beta > 0$ coins β for $\Delta_\alpha > 0$ coins α, must satisfy

$$R_\alpha - \Delta_\alpha = R_\alpha \left(\frac{R_\beta}{R_\beta + \Delta_\beta} \right)^\gamma \tag{1.1}$$

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After each transaction, the reserves are updated in the following way: $R_\alpha \mapsto R_\alpha - \Delta_\alpha$, $R_\beta \mapsto R_\beta + \Delta_\beta$, and $k \mapsto (R_\alpha - \Delta_\alpha)(R_\beta + \Delta_\beta)$. We will always require that $R_\alpha, R_\beta > 0$, such that any trade that results in a nonpositive reserve is never fulfilled (*i.e.*, has infinite cost).[1]

1.2 Path-independence of trades

Assume we have two possible trades with the contract. One is performed by trading some amount Δ_β to receive some amount of coin Δ_α, and a second is done by performing two trades which sum to the same amount, $\Delta_\beta = {\Delta_\beta}^1 + {\Delta_\beta}^2$, to receive some ${\Delta_\alpha}'$.

$$\Delta_\alpha = -R_\alpha \left(\left(\frac{R_\beta}{R_\beta + \Delta_\beta} \right)^\gamma - 1 \right) \tag{1.2}$$

Trading Δ_β always results in the same amount of coin as performing the two trades ${\Delta_\beta}^1, {\Delta_\beta}^2$ (i.e. $\Delta_\alpha = {\Delta_\alpha}'$):

$$\begin{aligned} {\Delta_\alpha}' = R_\alpha \left(1 - \left(\frac{R_\beta}{R_\beta + {\Delta_\beta}^1} \right)^\gamma \right) + R_\alpha \left(\left(\frac{R_\beta}{R_\beta + {\Delta_\beta}^1} \right)^\gamma - \left(\frac{R_\beta}{R_\beta + {\Delta_\beta}^1 + {\Delta_\beta}^2} \right)^\gamma \right) \\ {\Delta_\alpha}' = R_\alpha - R_\alpha \left(\frac{R_\beta}{R_\beta + {\Delta_\beta}^1 + {\Delta_\beta}^2} \right)^\gamma \\ \Delta'_\alpha = \Delta_\alpha. \end{aligned} \tag{1.3}$$

This property is not critical to the arguments which follow but is helpful to show that strategies will not depend on splitting up trades in a particular way (the limiting sum of a trade divided infinitely behaves as expected).

1.3 Optimal arbitrage in Uniswap

Coins α and β trade in a reference market and a Uniswap contract. α is a risky asset with reference market price m_p and β is riskless (cash).

We seek to maximize the profit made from trading some amount Δ_β of coin β to some amount Δ_α of coin α via the Uniswap market, resulting in profit $\Delta'_\beta - \Delta_\beta$ (marked to the reference market).

If our profit is positive (that is, if $\Delta'_\beta - \Delta_\beta > 0$), then we say that there is an *arbitrage* opportunity, since we have made money 'for free' (*i.e.*, by only trading coins within different markets), without taking on any risk. The optimal arbitrage problem then asks: what is the maximum profit that can be made by this scheme?

[1] Note that our invariant 1.1 is not used in practice by Uniswap. Dan Robinson derived it by solving a differential form in which the marginal price of any trade is discounted exactly by γ: $-\frac{\Delta_\beta}{\Delta_\alpha} = \gamma \frac{R_\alpha}{R_\beta}$. However, exponentiation is expensive so Uniswap implements its fee by multiplying γ into Δ_β. This approximates our "continuous" invariant for $\gamma \approx 1$, but does not share 1.1's path-independence.

In the infinitely liquid market case, *i.e.*, in the case that $\Delta'_\beta = m_p\Delta_\alpha$, we can phrase the optimal arbitrage problem as the following optimization problem.

$$\begin{array}{ll} \text{maximize} & m_p\Delta_\alpha - \Delta_\beta \\ \text{subject to} & \Delta_\alpha, \Delta_\beta \geq 0 \\ & R_\alpha - \Delta_\alpha = R_\alpha \left(\dfrac{R_\beta}{R_\beta + \Delta_\beta}\right)^\gamma, \end{array} \tag{1.4}$$

with optimization variables $\Delta_\alpha \in \mathbf{R}$ and $\Delta_\beta \in \mathbf{R}$. Here, $(1-\gamma)$ is the Uniswap exchange fee.

Optimality conditions. Note that a maximum of a concave function over an interval happens either at (a) the interior of an interval or (b) at its boundary. In the latter case, it is not hard to show that a maximum is attained at the point on the boundary closest to the unconstrained maximum.[2] Because of this, we only have to consider the unconstrained version of problem (1.4), over the interval $[0, +\infty)$.

In this case, the optimality conditions are the point for which the objective of (1.4) has zero derivative, which happens when

$$\Delta_\alpha = R_\alpha \left(1 - \left(\frac{m_p\gamma R_\alpha}{R_\beta}\right)^{\frac{1}{1+\gamma}-1}\right).$$

By the statement above, then the optimal solution to (1.4) is

$$\Delta_\alpha^\star = \left(R_\alpha \left(1 - \left(\frac{m_p\gamma R_\alpha}{R_\beta}\right)^{\frac{1}{1+\gamma}-1}\right)\right)_+,$$

where $(x)_+ = \max\{x, 0\}$ for $x \in \mathbf{R}$. The optimal $\Delta_\beta^\star$ can be easily derived using the $\Delta_\alpha^\star$ derived above and the constant product formula (1.1).

$$\Delta_\beta^\star = \left(R_\beta \left(\left(\frac{m_p\gamma R_\alpha}{R_\beta}\right)^{\frac{1}{1+\gamma}} - 1\right)\right)_+,$$

Now, note that $\Delta_\alpha^\star$ is zero if, and only if

$$R_\alpha \left(1 - \left(\frac{m_p\gamma R_\alpha}{R_\beta}\right)^{\frac{1}{1+\gamma}-1}\right) \leq 0 \iff m_u \leq \gamma^{-1} m_p,$$

where $m_u = R_\alpha / R_\beta$ is the marginal Uniswap market price of α without fees.

Since the objective of (1.4) is zero whenever there is no arbitrage opportunity (and the objective is only zero at $\Delta_\alpha^\star = 0$, by strict convexity), then the above implies there is no $\alpha \to \beta$ arbitrage in the presence of an infinitely liquid market. Swapping α for β yields the same result derived via no-arbitrage in the general case:

$$\gamma m_p \leq m_u \leq \gamma^{-1} m_p.$$

[2] The proof follows from the fact that a concave function over $\mathbf{R}$ is monotonically nondecreasing (nonincreasing) to the left (right) of its maximum.

2 No-Arbitrage Under Brownian Motion

Thus far, we have considered a static model of the system. We will now introduce a notion of time by replacing the fixed reference market price, m_p, with one that follows a geometric Brownian motion process, $S(t)$:

$$\frac{\delta S(t)}{S(t)} = \mu \delta t + \sigma \delta Z(t)$$

We can consider the evolution of the system over discretized timesteps δt by requiring the no-arbitrage conditions $\Delta_\alpha^\star$ and $\Delta_\beta^\star$ hold through the change in price $S(t)$ to $S(t+\delta t)$. Two common terms are pulled out into placeholders, C_h and C_l, for clarity:

$$C_l = \frac{\gamma R_\alpha(t) S(t+\delta t)}{R_\beta(t)}$$

$$C_h = \frac{R_\alpha(t) S(t+\delta t)}{\gamma R_\beta(t)}$$

$$R_\alpha(t+\delta t) = R_\alpha(t) \begin{cases} C_l^{-\frac{\gamma}{\gamma+1}} & \gamma S(t+\delta t) > \frac{R_\beta(t)}{R_\alpha(t)} \\ 1 & \gamma S(t+\delta t) \le \frac{R_\beta(t)}{R_\alpha(t)} \le \frac{1}{\gamma} S(t+\delta t) \\ C_h^{-\frac{1}{\gamma+1}} & \frac{1}{\gamma} S(t+\delta t) < \frac{R_\beta(t)}{R_\alpha(t)} \end{cases} \tag{2.1}$$

$$R_\beta(t+\delta t) = R_\beta(t) \begin{cases} C_l^{\frac{1}{\gamma+1}} & \gamma S(t+\delta t) > \frac{R_\beta(t)}{R_\alpha(t)} \\ 1 & \gamma S(t+\delta t) \le \frac{R_\beta(t)}{R_\alpha(t)} \le \frac{1}{\gamma} S(t+\delta t) \\ C_h^{\frac{\gamma}{\gamma+1}} & \frac{1}{\gamma} S(t+\delta t) < \frac{R_\beta(t)}{R_\alpha(t)} \end{cases} \tag{2.2}$$

where $R_\alpha(0) = R_\beta(0) = S(0) = 1$ and $0 \le \gamma < 1$.

2.1 Restatement

Special thanks to Prof. Steven Shreve for authoring this helpful restatement of the problem.

Reflection Behavior The key observation is that no profitable arbitrages exist in the "no-trade" region defined by the right side of the second terms in2.1 and 2.2:

$$\gamma S(t) \le \frac{R_\beta(t)}{R_\alpha(t)} \le \frac{1}{\gamma} S(t), \quad t \ge 0. \tag{2.3}$$

At times when one of the inequalities in (2.3) is tight, control ("reflection") is applied to prevent the inequality from being violated. An intuition is to imagine this mechanism as "continuous arbitrage:" infinitesimal trades that keep the ratio R_α / R_β from exceeding the lower or upper boundary. We must determine the direction of reflection for the pair $(R_\alpha(t), R_\beta(t))$ when it occurs on either end.

To do this, we observe from (2.1) and (2.2) that when $R_\beta(t)/R_\alpha(t)$ is at the upper boundary $S(t)/\gamma$:

$$\frac{R_\beta(t+\delta t)-R_\beta(t)}{R_\alpha(t+\delta t)-R_\alpha(t)}=\frac{R_\beta(t)}{R_\alpha(t)}*\frac{\left(\frac{R_\alpha(t)S(t+\delta t)}{\gamma R_\beta(t)}\right)^{\frac{\gamma}{\gamma+1}}-1}{\left(\frac{R_\alpha(t)S(t+\delta t)}{\gamma R_\beta(t)}\right)^{-\frac{1}{\gamma+1}}-1}. \tag{2.4}$$

We use the condition $\lim_{\delta t\downarrow 0} S(t+\delta t)=S(t)$ to evaluate the limit in (2.4) as $\delta t\downarrow 0$. L'Hôpital's rule implies:

$$\lim_{\delta t\downarrow 0}\frac{\left(\frac{R_\alpha(t)S(t+\delta t)}{\gamma R_\beta(t)}\right)^{\frac{\gamma}{\gamma+1}}-1}{\left(\frac{R_\alpha(t)S(t+\delta t)}{\gamma R_\beta(t)}\right)^{-\frac{1}{\gamma+1}}-1}=\lim_{x\to 1}\frac{x^{\frac{\gamma}{\gamma+1}}-1}{x^{-\frac{1}{\gamma+1}}-1}=\lim_{x\to 1}\frac{\frac{\gamma}{\gamma+1}x^{\frac{\gamma}{\gamma+1}-1}}{-\frac{1}{\gamma+1}x^{-\frac{1}{\gamma+1}-1}}=-\gamma.$$

Therefore,

$$\lim_{\delta t\downarrow 0}\frac{R_\beta(t+\delta t)-R_\beta(t)}{R_\alpha(t+\delta t)-R_\alpha(t)}=-\frac{\gamma R_\beta(t)}{R_\alpha(t)}. \tag{2.5}$$

At the upper boundary we have that $C_h^{-\frac{1}{\gamma+1}}>1$, and so, according to the third case in (2.1), $R_\alpha(t+\delta t)>R_\alpha(t)$. The reflection at the upper boundary $R_\beta(t)/R_\alpha(t)=S(t)/\gamma$ is in the $(R_\alpha(t),-\gamma R_\beta(t))$ direction in the (R_α,R_β)-plane.

At the lower boundary, when $R_\beta(t)/R_\alpha(t)=\gamma S(t)$, we have by a similar argument that

$$\begin{aligned}\lim_{\delta t\downarrow 0}\frac{R_\beta(t+\delta t)-R_\beta(t)}{R_\alpha(t+\delta t)-R_\alpha(t)}&=\lim_{\delta t\downarrow 0}\frac{R_\beta(t)}{R_\alpha(t)}*\frac{\left(\frac{\gamma R_\alpha(t)S(t+\delta t)}{R_\beta(t)}\right)^{\frac{1}{\gamma+1}}-1}{\left(\frac{\gamma R_\alpha(t)S(t+\delta t)}{R_\beta(t)}\right)^{-\frac{\gamma}{\gamma+1}}-1}\\&=\frac{R_\beta(t)}{R_\alpha(t)}\lim_{x\to 1}\frac{\frac{1}{\gamma+1}x^{\frac{1}{\gamma+1}-1}}{-\frac{\gamma}{\gamma+1}x^{\frac{\gamma}{\gamma+1}-1}}=-\frac{R_\beta(t)}{\gamma R_\alpha(t)}.\end{aligned} \tag{2.6}$$

And with $C_l^{-\frac{\gamma}{\gamma+1}}<1$, according to the first case in (2.1), $R_\alpha(t+\delta t)<R_\alpha(t)$. The reflection at the lower boundary $R_\beta(t)/R_\alpha(t)=\gamma S(t)$ is in the $(-\gamma R_\alpha(t),R_\beta(t))$ direction in the (R_α,R_β)-plane.

Restatement We are now ready to restate the problem. We have a geometric Brownian motion

$$dS(u)=\mu S(u)du+\sigma S(u)dZ(u),\quad S(0)=1, \tag{2.7}$$

where Z is a standard Brownian motion. There are two nondecreasing continuous processes L and U with $L(0)=U(0)=0$. The process L grows only when $R_\beta(t)/R_\alpha(t)=\gamma S(t)$ and the process U grows only when $R_\beta(t)/R_\alpha(t)=S(t)/\gamma$, i.e.

$$L(t) = \int_0^t \mathbb{1}_{\{R_\beta(u)/R_\alpha(u) = \gamma S(u)\}} dL(u), \quad U(t) = \int_0^t \mathbb{1}_{\{R_\beta(u)/R_\alpha(u) = S(u)/\gamma\}} dU(u), \quad t \geq 0. \quad (2.8)$$

We have two finite-variation continuous processes R_α and R_β given for $t \geq 0$ by

$$\begin{aligned} R_\alpha(t) &= 1 - \gamma \int_0^t R_\alpha(u) dL(u) + \int_0^t R_\alpha(u) dU(u), \\ R_\beta(t) &= 1 + \int_0^t R_\beta(u) dL(u) - \gamma \int_0^t R_\beta(u) dU(u). \end{aligned} \quad (2.9)$$

We believe that for each fixed $\gamma \in (0, 1)$, the processes L, U, R_α and R_β exist and are uniquely determined by these conditions.

3 Open Problems

The value of the LP portfolio at time t is the value of coins α and β held at the time. The wealth process $W(t)$ is simply:

$$W(t) = R_\alpha(t) S(t) + R_\beta(t). \quad (3.1)$$

3.1 Growth Rate of LP Wealth

Given known μ, σ, and γ what is the (asymptotic) exponential growth rate of LP wealth G:

$$G = \mathbb{E}\left[\lim_{T\to\infty} \frac{1}{T} \log(W(T))\right]? \quad (3.2)$$

3.2 Optimal Fee

Given known μ and σ, what is the fee γ which maximizes the growth rate of LP wealth $G^\star$:

$$G^\star = \max_{0<\gamma<1} \left(\mathbb{E}\left[\lim_{T\to\infty} \frac{1}{T} \log(W(T))\right]\right)? \quad (3.3)$$

3.3 Excess Return

Are there conditions on μ and σ for which the optimal growth rate of LP wealth $G^\star$ can meet or exceed that of an unrebalanced portfolio? [3].

$$G^\star - \max\left(\mu - \frac{\sigma^2}{2}, 0\right) \geq 0? \quad (3.4)$$

[3] The growth rate of which is the maximum of the constituent assets

# 4 Acknowledgements

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